

Scaling limit of the local time of the $(1, L)$ –random walk¹

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Abstract

It is well known (Donsker’s Invariance Principle) that the random walk converges to Brownian motion by scaling. In this paper, we will prove that the scaled local time of the $(1, L)$ –random walk converges to that of the Brownian motion. The results was proved by Rogers (1984) in the case $L = 1$. Our proof is based on the intrinsic multiple branching structure within the $(1, L)$ –random walk revealed by Hong and Wang (2013).

Keywords: random walk, multi-type branching process, local time, Brownian motion.

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1 Introduction and Main Results

Donsker’s Invariance Principle tells us that the random walk converge to the Brownian motion by proper space and time scaling. It is naturally to consider the scaling limit of the local times. One can not get it directly from the continuous theorem because the local time is not a continuous function of the Brownian motion. For the simple random walk (i.e., $L = 1$, the nearest random walk), Rogers ([7],1984) confirmed the result based on the branching structure within the simple symmetric random walk first introduced by Dwass ([2], 1975) and the convergence of the scaling branching processes demonstrated by Lamperti ([4], 1967) and Lindvall ([5],1972), combining the Ray-Knight Theorem. In the present paper we consider the $(1, L)$ –random walk. At first we can express the local time as a linear function of the intrinsic multi-type branching processes within the $(1, L)$ –random walk, which has been revealed by Hong and Wang ([3], 2013) recently. After that, We obtain the scaling limit following the usual schedule by proving the convergence of the finite dimensional distribution and the tightness.

We consider the $(1, L)$ – random walk on the half line reflected at 0 , i.e., a Markov chain $\{X_n\}_{n \geq 0}$ on $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$ with $X_0 = 0$ and the transition probabilities specified by, for $n \geq 0, i \geq 1$,

$$P(X_{n+1} = 1 | X_n = 0) = 1$$

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$$P(X_{n+1} = i + l | X_n = i) = \begin{cases} p_l, & \text{for } l = 1, \dots, L, \\ q, & \text{for } l = -1. \end{cases} \quad (1.1)$$

where $p_1 + p_2 + \dots + p_L + q = 1$, $0 < p_1, p_2, \dots, p_L, q < 1$ and “symmetric”

$$EX_n = p_1 + 2p_2 + \dots + L \cdot p_L - q = 0. \quad (1.2)$$

Obviously, this Markov chain is irreducible and recurrent. For any position $j \geq 0$, the *local time* $L(j; n)$ at j is defined as the visiting number by the Markov chain $\{X_n\}_{n \geq 0}$ before n ,

$$L(j; n) = \#\{0 \leq r \leq n : X_r = j\} \quad \text{for } j, n \geq 0. \quad (1.3)$$

Define the excursion time at position 0, $\tau_0 = 0$, and for $n \geq 1$,

$$\tau_n = \inf\{k > \tau_{n-1} : X_k = 0\}. \quad (1.4)$$

We are interested in the local time $L(j; \tau_N)$, the visiting number at position j by $\{X_n\}_{n \geq 0}$ in the first N excursions. Define the scaling $l_N(x)$ as follows, for $\forall N \in \mathbb{Z}^+, x \in [0, 1]$,

$$l_N(x) = \begin{cases} \frac{L([Nx]; \tau_N)}{N}, & \text{for } Nx \geq 1, \\ 2/\sigma^2, & \text{for } 0 \leq Nx < 1, \end{cases} \quad (1.5)$$

where $\sigma^2 = DX_n = p_1 + q + 4p_2 = 6q - 2$, $n > 1$. Our main results is the following

Theorem 1.1. *As the random elements on $D[0, \infty)$,*

$$l_N(\cdot) \Rightarrow H(\cdot) \quad (1.6)$$

where $H(x), x \in [0, \infty)$ is a diffusion processes which is the solution of the stochastic differential equation

$$H(x) = \frac{2}{\sigma^2} + \frac{2}{\sigma} \int_0^x (H(s)^+)^{\frac{1}{2}} dB_s, \quad (1.7)$$

where $B(t), t \in [0, \infty)$ is standard Brownian motion.

Remark 1.1 *Actually $(H(t))_{t \geq 0}$ is a continuous (time and states) branching process with $H(0) = \frac{2}{\sigma^2}$ and its transition probabilities satisfy, for $\lambda \geq 0$,*

$$\int e^{-\lambda y} p_t(x, dy) = \exp\left(-x \frac{\lambda}{1 + \frac{2}{\sigma^2} \lambda t}\right). \quad (1.8)$$

We will prove the Theorem following the usual schedule by proving the convergence of the finite dimensional distribution and the tightness. \square

Remark 1.2 *Let $\{B(t) : t \geq 0\}$ be Brownian motion on \mathbf{R} , $B_0 = 0$, and let $l(x, t)$ be its local time. Define $T = \inf\{t : l(0, t) > 1\}$. Ray-Knight Theorem tells us that*

$$(l(x, T))_{x \geq 0} = (Z_t)_{t \geq 0},$$

where Z is the solution of the stochastic differential equation

$$Z_t = 1 + \sqrt{2} \int_0^t (Z_s^+)^{\frac{1}{2}} dB_s,$$

and its transition probabilities satisfy, for $\lambda \geq 0$,

$$\int e^{-\lambda y} p_t(x, dy) = e^{-x \frac{\lambda}{1+\lambda t}}.$$

From this point of view, (1.6) can be rewritten as

$$l_N(\cdot) \Rightarrow \frac{1}{\sigma^2} l^*(\cdot, T) \quad (1.9)$$

where $l^*(x, T) = l(x, T) + l(-x, T)$, $x \geq 0$, which coincides with the scaling behavior of the $(1, L)$ -random walk $\frac{X_{[nx]}}{\sqrt{n}} \rightarrow B_{\sigma^2 x}$. \square

2 Local time and branching process in the $(1, L)$ - random walk

We will express the local time in terms of the intrinsic branching structure within the $(1, L)$ -random walk revealed by Hong and Wang ([3], 2013). For simplicity of the notation, in what follows, we will restrict ourselves to consider the case $L = 2$.

2.1 Branching process within the $(1, L)$ -random walk

Let us recall the 2-type branching processes within the $(1, 2)$ - random walk ([3], 2013). For $i \geq 1$, to record the visiting number at the position i by the walk $\{X_n\}_{n \geq 0}$ in the first excursion at 0, define

$$\begin{aligned} U_1(i-1) &= \#\{0 \leq n < \tau_1 : X_n < i, X_{n+1} = i\}, \\ U_2(i-1) &= \#\{0 \leq n < \tau_1 : X_n < i, X_{n+1} = i+1\}. \end{aligned} \quad (2.1)$$

For $n \geq 0$, let $U_n = (U_1(n), U_2(n))$.

Theorem 2.1. (Hong & Wang, [3], 2013) (1) The process $\{U_n\}_{n=0}^\infty$ is a 2-type critical branching process whose branching mechanism is given by $P(U_0 = (1, 0)) = 1$, and for $k \geq 1$

$$\begin{aligned} P^{(1)}(u_1, u_2) &:= P(U_{k+1} = (u_1, u_2) | U_k = e_1) = \frac{(u_1 + u_2)!}{u_1! u_2!} p_1^{u_1} p_2^{u_2} q, \\ P^{(2)}(u_1, u_2) &:= P(U_{k+1} = (u_1 + 1, u_2) | U_k = e_2) = \frac{(u_1 + u_2)!}{u_1! u_2!} p_1^{u_1} p_2^{u_2} q. \end{aligned} \quad (2.2)$$

(2) Let $m_{ij} = E(U_j(n+1) | U_n = e_i)$ be the mean offspring of type j particles born from a single type i parent particle. Define the mean offspring matrix $M = \{m_{ij}, i, j = 1, 2\}$, then

$$M = \begin{pmatrix} \rho_1 & \rho_2 \\ 1 + \rho_1 & \rho_2 \end{pmatrix} \quad (2.3)$$

where $\rho_i = \frac{p_i}{q}$, $i = 1, 2$.

Remark 2.1. This is followed from the result of branching structure in the $(L, 1)$ random walk which is revealed by Hong and Wang([3], 2013). A little bit attention should be noted is here we view the branching structure from the “upward” direction whereas in Hong and Wang([3], 2013) from the “downward” direction, because here we consider the reflected random walk. We should consider here the branching processes $\{U_n\}_{n \geq 0}$ begin at $n \geq 1$ and with $U_0 = (1, 0)$ as the “immigration”. In addition, the “symmetric” condition (1.2) ensures the maximal eigenvalue of the offspring matrix M is 1, i.e., the multitype branching process $\{U_n\}_{n \geq 0}$ is critical.

With Theorem 2.1 in hand, we can calculate the probability generating function of $\{U_n\}_{n \geq 0}$ as follows, which is useful in the proof of the scaling limit.

Proposition 2.1. (1) Denote $\mathbf{s} = (s_1, s_2)$, $g^{(i)}(s_1, s_2) := E(\mathbf{s}^{U_i} | U_1 = e_i)$, $i = 1, 2$; then

$$\begin{aligned} g^{(1)}(s_1, s_2) &= \frac{q}{1 - p_1 s_1 - p_2 s_2} \\ g^{(2)}(s_1, s_2) &= \frac{q s_1}{1 - p_1 s_1 - p_2 s_2} \end{aligned} \quad (2.4)$$

(2) Let $f_n(s_1, s_2)$ be the generating function of $\{U_n = (U_1(n), U_2(n))\}_{n=0}^\infty$, i.e., $f_n(s_1, s_2) := E(\mathbf{s}^{U_n} | U_0 = e_1)$, we have for $n \geq 1$

$$f_n(s_1, s_2) = \frac{1 + (1 - s_1)a_{n-1} + (1 - s_2)b_{n-1}}{1 + (1 - s_1)a_n + (1 - s_2)b_n} \quad (2.5)$$

where $(a_0, b_0) = (0, 0)$, for $n > 0$, $(a_n, b_n) = \mathbf{u}(M^{n-1} + M^{n-2} + \cdots + M + I)$ and $\mathbf{u} = (\rho_1, \rho_2)$.

Proof. (1) By direct calculation from the branching mechanism (2.2),

$$\begin{aligned} g^{(1)}(s_1, s_2) &= \sum_{u_1, u_2=0}^{\infty} P^{(1)}(u_1, u_2) s_1^{u_1} s_2^{u_2} \\ &= \sum_{u_1, u_2=0}^{\infty} \frac{(u_1 + u_2)!}{u_1! u_2!} p_1^{u_1} p_2^{u_2} q s_1^{u_1} s_2^{u_2} \\ &= \frac{q}{1 - p_1 s_1 - p_2 s_2} \\ &= \frac{1}{1 + (1 - s_1)\rho_1 + (1 - s_2)\rho_2}, \end{aligned}$$

and similarly to get $g^{(2)}(s_1, s_2)$.

(2) We will show (2.5) by induction. Firstly, when $n = 1$,

$$\begin{aligned} f_1(\mathbf{s}) = g^{(1)}(s_1, s_2) &= \frac{1}{1 + (1 - s_1)\rho_1 + (1 - s_2)\rho_2} \\ &= \frac{1 + (1 - s_1)a_0 + (1 - s_2)b_0}{1 + (1 - s_1)a_1 + (1 - s_2)b_1}. \end{aligned}$$

Assume (2.5) is true when $k \leq n - 1$, we have

$$f_n(\mathbf{s}) = f_{n-1}(g^{(1)}(s_1, s_2), g^{(2)}(s_1, s_2))$$

$$\begin{aligned}
&= \frac{1 + (1 - \frac{1}{1+(1-s_1)\rho_1+(1-s_2)\rho_2})a_{n-2} + (1 - \frac{s_1}{1+(1-s_1)\rho_1+(1-s_2)\rho_2})b_{n-2}}{1 + (1 - \frac{1}{1+(1-s_1)\rho_1+(1-s_2)\rho_2})a_{n-1} + (1 - \frac{s_1}{1+(1-s_1)\rho_1+(1-s_2)\rho_2})b_{n-1}} \\
&= \frac{1 + (1 - s_1)(\rho_1 + \rho_1 a_{n-2} + \rho_1 b_{n-2} + b_{n-2}) + (1 - s_2)(\rho_2 + \rho_2 a_{n-2} + \rho_2 b_{n-2})}{1 + (1 - s_1)(\rho_1 + \rho_1 a_{n-1} + \rho_1 b_{n-1} + b_{n-1}) + (1 - s_2)(\rho_2 + \rho_2 a_{n-1} + \rho_2 b_{n-1})} \\
&= \frac{[(a_{n-2}, b_{n-2})M + (\rho_1, \rho_2)](1 - s_1, 1 - s_2)'}{[(a_{n-1}, b_{n-1})M + (\rho_1, \rho_2)](1 - s_1, 1 - s_2)'} \\
&= \frac{1 + (1 - s_1)a_{n-1} + (1 - s_2)b_{n-1}}{1 + (1 - s_1)a_n + (1 - s_2)b_n}
\end{aligned}$$

and $(a_n, b_n) = (a_{n-1}, b_{n-1})M + (\rho_1, \rho_2)$ for $n \geq 1$. \square

Remark 2.2 It should be better to write $f_n^{(1)}(s_1, s_2)$ for $f_n(s_1, s_2)$ because $f_n(s_1, s_2) := E(\mathbf{s}^{U_n} | U_0 = \mathbf{e}_1)$. In next section, we need the other one $f_n^{(2)}(s_1, s_2) := E(\mathbf{s}^{U_n} | U_0 = \mathbf{e}_2)$. By the similar calculation, we have $f_1^{(2)}(s_1, s_2) = g^{(2)}(s_1, s_2)$, and for $n \geq 2$,

$$f_n^{(2)}(s_1, s_2) = \frac{1 + (1 - s_1)a_{n-2} + (1 - s_2)b_{n-2}}{1 + (1 - s_1)a_n + (1 - s_2)b_n}. \quad (2.6)$$

\square

2.2 Local time $L(j; \tau_N)$

From the definition of $U_n = (U_1(n), U_2(n))$ in (2.1), we can easily express the local time $L(j; \tau_N)$ in terms of the 2-type branching processes $\{U_n\}_{n \geq 0}$ as follows,

Theorem 2.2. (1) For $j \geq 1$,

$$L(j; \tau_1) = U_1(j - 1) + U_1(j) + U_2(j). \quad (2.7)$$

(2) For any positive integral N ,

$$L(j; \tau_N) = \sum_{r=1}^N \xi_r, \quad (2.8)$$

where ξ_r , $r = 1, 2, \dots$ are i.i.d. random variables, distributed as $L(j; \tau_1)$.

Proof (1) The local time $L(j; \tau_1)$ is the visiting number at position j by the trajectory of the $(1, 2)$ -random walk within the first excursion at 0 (i.e., between $0 \leq n \leq \tau$), it is the summation of two kind steps: “upper steps” (visits at position j from below j) and “down steps” (visits at position j from above j). By the definition (2.1), the “upper steps” is just $U_1(j - 1)$; with regard the recurrence of the $(1, 2)$ -walk and the walk downs step by step, the “down steps” to j equals to the steps from (and below) j to $j + 1$ (which is $U_1(j)$) plus the steps from j to $j + 2$ (which is $U_2(j)$); and so (2.7) is followed.

(2) Decompose the trajectory of the $(1, 2)$ -random walk, $L(j; \tau_N)$ is the summation of the visiting numbers at j in N excursions, which are the i.i.d. random variables. For more details, write $L(j; m, n) := \{m \leq r \leq n : X_r = j\}$, we have

$$L(j; \tau_N) = \sum_{r=1}^N L(j; \tau_{r-1}, \tau_r). \quad (2.9)$$

Write $\xi_r := L(j; \tau_{r-1}, \tau_r)$, then $\xi_r, r = 1, 2, \dots$ are i.i.d. random variables, distributed as $L(j; \tau_1)$ by the Markov property of the (1, 2)-random walk. \square

3 Proof of Theorem 1.1

With the explicit expression of the local time (2.8) in terms of the multi-type branching process $\{U_n\}_{n \geq 0}$, we are now at the position to prove the main result. Firstly, note that as in the (1) of Theorem 2.2, we have

$$\xi_r := L(j; \tau_{r-1}, \tau_r) = U_1^{(r)}(j-1) + U_1^{(r)}(j) + U_2^{(r)}(j), \quad (3.1)$$

where $\{U_n^{(r)} = (U_1^{(r)}(n), U_2^{(r)}(n)); n \geq 0\}, r = 1, 2, \dots$ are i.i.d., distributed as $\{(U_1(n), U_2(n)); n \geq 0\}$. Actually, for each $r \geq 1$, $\{U_n^{(r)}; n \geq 0\}$ is a 2-type branching processes corresponding the r^{th} excursion at 0 of the random walk, which is independent of each other and with the same branching mechanism with $\{U_n = (U_1(n), U_2(n)); n \geq 0\}$. Recall (1.5) the scaling of the local time $L(j; \tau_N)$, by (2.9) and (3.1),

$$\begin{aligned} l_N(x) &= \frac{L([Nx]; \tau_N)}{N} = \sum_{r=1}^N \frac{L([Nx]; \tau_{r-1}, \tau_r)}{N} \\ &= \sum_{r=1}^N \frac{U_1^{(r)}([Nx]-1) + U_1^{(r)}([Nx]) + U_2^{(r)}([Nx])}{N} \\ &:= U_{N,1}(x - \frac{1}{N}) + U_{N,1}(x) + U_{N,2}(x), \end{aligned} \quad (3.2)$$

where we write, for $i = 1, 2$,

$$U_{N,i}(x) = \sum_{r=1}^N \frac{U_i^{(r)}([Nx])}{N}. \quad (3.3)$$

So in what follows, we just need to consider the weak convergence of $\{2U_{N,1}(x) + U_{N,2}(x); x \geq 0\}$, with regard of the strong convergence to 0 of $U_{N,1}(x - \frac{1}{N}) - U_{N,1}(x)$ by the strong law of large numbers. Nakagawa ([6], 1986) considered the convergence of critical multitype Galton-Watson branching processes, which generalized the results of Lindvall ([5], 1972) to the multitype case. However, here we can not apply the result directly, because our target $\{2U_{N,1}(x) + U_{N,2}(x); x \geq 0\}$ is different from the $\{\hat{Y}_n(t); t \geq 0\}$ in Theorem 1.1 of ([6], 1986). We can calculate explicitly based on the branching structure Theorem 2.1 to specify the role of the σ^2 in (1.7) and (1.8) comparing with Theorem 1.1 of Nakagawa ([6], 1986).

Step 1 Warm up: the convergence of one dimensional distribution

Lemma 3.1. *For $x \in [0, 1]$, as $N \rightarrow \infty$*

$$2U_{N,1}(x) + U_{N,2}(x) \Rightarrow H(x),$$

where the laplace transform of $H(x)$ is given by

$$\Phi(x, \lambda) = \exp\left(-\frac{\frac{2}{\sigma^2}\lambda}{1 + \frac{2}{\sigma^2}x\lambda}\right),$$

and $\sigma^2 = DX_n = p_1 + q + 4p_2 = 6q - 2$, $n > 1$.

Proof Recall the notation (3.3), we calculate the Laplace transformation of $2U_{N,1}(x) + U_{N,2}(x)$,

$$\begin{aligned}
 F_{2U_{N,1}(x)+U_{N,2}(x)}(\lambda) &= Eexp[-\lambda(2U_{N,1}(x) + U_{N,2}(x)|U_0 = \mathbf{e}_1)] \\
 &= \left\{ Eexp \left[-\frac{\lambda}{N} [(2U_1([Nx]) + U_2([Nx]))] \right] \right\}^N \\
 &= \left[f_{[Nx]}(e^{-\frac{2\lambda}{N}}, e^{-\frac{\lambda}{N}}) \right]^N \\
 &= \left[\frac{1 + (a_{[Nx]-1}, b_{[Nx]-1})\mathbf{v}'}{1 + (a_{[Nx]}, b_{[Nx]})\mathbf{v}'} \right]^N.
 \end{aligned} \tag{3.4}$$

the last step is from (2.5), where $\mathbf{v} = (1 - e^{-\frac{2\lambda}{N}}, 1 - e^{-\frac{\lambda}{N}})$, $(a_n, b_n) = \mathbf{u}(M^{n-1} + M^{n-2} + \dots + M + I)$ and $\mathbf{u} = (\rho_1, \rho_2)$. If we denote

$$\begin{cases} A_N(x) := (a_{[Nx]-1}, b_{[Nx]-1})\mathbf{v}', \\ B_N(x) := (a_{[Nx]}, b_{[Nx]})\mathbf{v}'. \end{cases} \tag{3.5}$$

(3.4) can be written as

$$\begin{aligned}
 F_{2U_{N,1}(x)+U_{N,2}(x)}(\lambda) &= \left(\frac{1 + A_N(x)}{1 + B_N(x)} \right)^N \\
 &= \left[\left(1 + \frac{A_N(x) - B_N(x)}{1 + B_N(x)} \right)^{(1+B_N(x))/(A_N(x)-B_N(x))} \right]^{N(A_N(x)-B_N(x))/(1+B_N(x))}.
 \end{aligned} \tag{3.6}$$

We are now to consider the limit of $A_N(x)$, $B_N(x)$ and $N(A_N(x) - B_N(x))$. To this end, recall M is the mean offspring matrix, see (2.3). Recall that in our “symmetric” model, we have $p_1 + p_2 + q = 1$, and $EX_n = p_1 + 2p_2 - q = 0$, $n \geq 1$. By calculation, two of the eigenvalues of M are 1 and $\alpha = \frac{1-2q}{q}$ (and $|\alpha| < 1$); and there is a matrix T

$$T = \begin{pmatrix} 1 & 1-2q \\ 2 & 1-q \end{pmatrix}, \quad T^{-1} = \frac{1}{3q-1} \begin{pmatrix} 1-q & 2q-1 \\ -2 & 1 \end{pmatrix} \tag{3.7}$$

such that

$$M = T \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} T^{-1}. \tag{3.8}$$

Then

$$\begin{aligned}
 A_N(x) &:= (a_{[Nx]-1}, b_{[Nx]-1})\mathbf{v}' = \mathbf{u}(M^{[Nx]-2} + M^{[Nx]-3} + \dots + M + I)\mathbf{v}' \\
 &= \mathbf{u}T \begin{pmatrix} [Nx]-1 & 0 \\ 0 & \alpha^{[Nx]-2} + \dots + \alpha + 1 \end{pmatrix} T^{-1}\mathbf{v}' \\
 &= \frac{1}{3q-1} \left(1, \frac{(2q-1)^2}{q} \right) \begin{pmatrix} [Nx]-1 & 0 \\ 0 & \alpha^{[Nx]-2} + \dots + \alpha + 1 \end{pmatrix} \begin{pmatrix} ((q-1)e^{-\frac{\lambda}{N}} - q)(e^{-\frac{\lambda}{N}} - 1) \\ (2e^{-\frac{\lambda}{N}} + 1)(e^{-\frac{\lambda}{N}} - 1) \end{pmatrix}
 \end{aligned}$$

$$\longrightarrow \lambda x \frac{2}{\sigma^2}, \quad (3.9)$$

as $N \rightarrow \infty$. Similarly, we get

$$B_N(x) := (a_{[Nx]}, b_{[Nx]})\mathbf{v}' = \mathbf{u}(M^{[Nx]-1} + M^{[Nx]-2} + \dots + M + I)\mathbf{v}' \longrightarrow \lambda x \frac{2}{\sigma^2}. \quad (3.10)$$

Obviously, $B_N(x) - A_N(x) = \mathbf{u}M^{[Nx]-1}\mathbf{v}'$, and

$$N(B_N(x) - A_N(x)) = N(\mathbf{u}M^{[Nx]-1}\mathbf{v}') \longrightarrow \lambda \frac{2}{\sigma^2}. \quad (3.11)$$

Combining (3.9)-(3.11) with (3.6), we get

$$\lim_{N \rightarrow \infty} F_{2U_{N,1}(x) + U_{N,2}(x)}(\lambda) = \exp\left(-\frac{\frac{2}{\sigma^2}\lambda}{1 + \frac{2}{\sigma^2}x\lambda}\right)$$

complete the proof. \square

Remark 3.1 We write $F_N^{(1)}(x; \lambda) := F_{2U_{N,1}(x) + U_{N,2}(x)}(\lambda) = \text{Exp}[-\lambda(2U_{N,1}(x) + U_{N,2}(x)) | U_0 = \mathbf{e}_1]$. In next step, we need the other one $F_N^{(2)}(x; \lambda) := \text{Exp}[-\lambda(2U_{N,1}(x) + U_{N,2}(x)) | U_0 = \mathbf{e}_2]$. By the similar calculation (with regard of (2.6)), we can get

$$\lim_{N \rightarrow \infty} F_N^{(2)}(x; \lambda) = \lim_{N \rightarrow \infty} F_N^{(1)}(x; \lambda) = \Phi(x, \lambda). \quad (3.12)$$

\square

Step 2 The convergence of finite dimensional distributions

Lemma 3.2. For $1 \leq i \leq k, \lambda_i \geq 0$; and $0 \leq x_1 \leq x_2 \leq \dots \leq x_k \leq 1$, we have, as $N \rightarrow \infty$

$$E\left(\exp\left\{-\sum_{i=1}^k \lambda_i(2U_{N,1}(x_i) + U_{N,2}(x_i))\right\}\right) \rightarrow E\left(\exp\left\{-\sum_{i=1}^k \lambda_i H(x_i)\right\}\right), \quad (3.13)$$

where $\{H(x), x \in [0, 1]\}$ is a diffusion, continuous (time and states) branching process with $H(0) = \frac{2}{\sigma^2}$ given in (1.7).

Proof. First of all, note that $\{H(x), x \in [0, 1]\}$ is a continuous (time and states) branching process whose Laplace transforms satisfy

$$E\left(\exp\left\{-\sum_{i=1}^k \lambda_i H(x_i)\right\}\right) = E\left(\exp\left\{-\sum_{i=1}^{k-1} \lambda_i H(x_i)\right\} \Phi^{H(x_{k-1})}(x_k - x_{k-1}, \lambda_k)\right). \quad (3.14)$$

We will prove (3.13) by induction. Firstly, Lemma (3.1) convince (3.13) for $k = 1$. Assume (3.13) is right when $k \leq m$, we will check (3.13) for $k = m + 1$.

$$E(\exp\{-\sum_{i=1}^{m+1} \lambda_i(2U_{N,1}(x_i) + U_{N,2}(x_i))\})$$

$$\begin{aligned}
&= E(E(\exp\{-\sum_{i=1}^{m+1} \lambda_i(2U_{N,1}(x_i) + U_{N,2}(x_i))\} | U_N(x_m))) \\
&= E\left(\exp\{-\sum_{i=1}^m \lambda_i(2U_{N,1}(x_i) + U_{N,2}(x_i))\} E(\exp\{-\lambda_{m+1}(2U_{N,1}(x_{m+1}) + U_{N,2}(x_{m+1}))\} | U_N(x_m))\right),
\end{aligned}$$

in which with the notation $F_N^{(1)}(x; \lambda)$ and $F_N^{(2)}(x; \lambda)$ in Remark 3.1,

$$\begin{aligned}
&E(\exp\{-\lambda_{m+1}(2U_{N,1}(x_{m+1}) + U_{N,2}(x_{m+1}))\} | U_N(x_m)) \\
&= [F_N^{(1)}(x_{m+1} - x_m, \lambda_{m+1})]^{2U_{N,1}(x_m)} [F_N^{(2)}(x_{m+1} - x_m, \lambda_{m+1})]^{U_{N,2}(x_m)}.
\end{aligned}$$

then, (3.12) and the induction for $k \leq m$ enable us to conclude that

$$\begin{aligned}
&E(\exp\{-\sum_{i=1}^{m+1} \lambda_i(2U_{N,1}(x_i) + U_{N,2}(x_i))\}) \\
&\longrightarrow E[\exp(-\sum_{i=1}^m \lambda_i H(i)) \Phi^{H(x_m)}(x_{m+1} - x_m, \lambda_{m+1})],
\end{aligned}$$

which is (3.13) for $k = m + 1$ with regard of (3.14). \square

Step 3 completing the proof of Theorem 1.1

We follow the standard schedule. The tool for proving the weak convergence of $\{2U_{N,1}(x) + U_{N,2}(x)\}$ to $\{H(x)\}$ is Theorem 13.5 in Billingsley ([1], 1999) which states that if $\{V_n\}_{n \geq 1}$, V are random elements in $D[0, 1]$ and

- (a) the finite dimensional distributions of V_n converge to those of V ;
- (b) the probability for jumps of V in the point 1 is zero;
- (c) there exist $\gamma \geq 0$, $\alpha > 1$ and F is continuous and non-decreasing on $[0, 1]$, such that, $E[|V_n(t_2) - V_n(t)|^\gamma \cdot |V_n(t) - V_n(t_1)|^\gamma] \leq |F(t_2) - F(t_1)|^\alpha$, holds for all n and all $0 \leq t_1 \leq t \leq t_2 \leq 1$;

then $V_n \Rightarrow V$ on $D[0, 1]$.

Part (a) is already established for $\{2U_{N,1} + U_{N,2}\}_{N \geq 1}$ in step 2. Since the processes H has continuous paths, (b) is also no problem. We just need to manage (c). To this end, we can follow Nakagawa ([6], 1986) almost line by line with some modifications as following.

From (2.1) we know $\{U_n\}$ is critical branching process satisfying the condition in ([6], 1986), so the second moments of U_n has similar asymptotic behavior, i.e.

$$d_{ij}(n) = E[U_i(n)U_j(n)] = (U_0 \cdot \boldsymbol{\mu})Q_2[\boldsymbol{\mu}]\nu_i\nu_jn + o(n) \text{ as } n \rightarrow \infty, (i, j = 1, 2)$$

where

$$Q_2[\boldsymbol{\mu}] = \boldsymbol{\nu} \cdot \mathbf{q}_2[\boldsymbol{\mu}] \text{ and } (\mathbf{q}_2[\boldsymbol{\mu}])_i = \sum_{j=1}^2 \sum_{k=1}^2 \mu_j b_{jk}^{(i)} \mu_k, \quad i = 1, 2,$$

with $\boldsymbol{\mu} = (\mu_1, \mu_2)'$, $\boldsymbol{\nu} = (\nu_1, \nu_2)$ is the right and left eigenvectors corresponding to the eigenvalue 1 of M in (2.3), and $\boldsymbol{\nu} \cdot \boldsymbol{\mu} = 1$, $\mathbf{1} \cdot \boldsymbol{\mu} = 1$. $b_{jk}^{(i)} = E\{U_j(1)U_k(1) | U(0) = \mathbf{e}_i\} - E\{U_j(1) | U_0 = \mathbf{e}_i\}E\{U_k(1) | U_0 = \mathbf{e}_i\}$. For our model, as (3.3) in Nakagawa([6], 1986), one has

$$E[(2U_1(n) + U_2(2))^2] = K_1(U_0 \cdot \boldsymbol{\mu})Q_2[\boldsymbol{\mu}]n + o(n) \text{ as } n \rightarrow \infty \quad (3.15)$$

with $K_1 = (2\nu_1 + \nu_2)^2$; and as a consequence, corresponding (3.5) in Nakagawa([6], 1986) we get, for arbitrary integers $m > n \geq 0$

$$\begin{aligned}
& E[(U_m \cdot \mathbf{a} - U_n \cdot \mathbf{a})^2 | U_n] \\
&= E[(U_m \cdot \mathbf{a})^2 | U_n] + (U_n \cdot \mathbf{a})^2 - 2(U_n \cdot \mathbf{a})(E[U_m \cdot \mathbf{a} | U_n]) \\
&= K_1(U_n \cdot \boldsymbol{\mu})Q_2[\boldsymbol{\mu}](m - n) + o(m - n) + (U_n \cdot \mathbf{a})^2 - 2(U_n \cdot \mathbf{a})(U_n \cdot M^{m-n} \cdot \mathbf{a}) \\
&\leq K_1(U_n \cdot \boldsymbol{\mu})Q_2[\boldsymbol{\mu}](m - n),
\end{aligned} \tag{3.16}$$

the last inequality holds because by calculation $2M^{m-n} \cdot \mathbf{a} > \mathbf{a}$. This is enough to get (c) as Nakagawa([6], 1986), and the proof is finished for the convergence in the $D[0, 1]$. It is easy to extend the convergence in $D[0, N]$ for any positive integer N by the scaling property of the local time of Brownian motion, and then in $D[0, \infty)$. \square

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